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Geometric phase of a qubit in dephasing environments

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Abstract

The geometric phase of a qubit coupled dephasingly to ohmic or non-ohmic bosonic environment exhibits a strong dependence on both an initial state of the system and spectral properties of the environment. Analysing the exact reduced dynamics we show that the geometric phase can be more easily maintained for the subohmic and ohmic environments than for the superohmic environment. We show the existence of the initial states for which the geometric phase is robust against coupling to the environment.

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1. Introduction

The geometric phase (GP) of a quantum state belongs to one of the fundamental concepts deeply integrated with the structure of quantum mechanics. The concept of GP has undergone a long evolution from the classical Pancharatnam notion and quantum Berry phase factor in the adiabatic and cyclic unitary evolution of non-degenerate states. Numerous generalizations have included nonadiabatic, noncyclic and nonunitary evolution, also for degenerate states. A great challenge is to study GPs in quantum open systems and for mixed states. This is motivated by the fact that all realistic systems are coupled to their environments. The potential application in quantum-information processing makes GPs attractive for geometric or holonomic quantum computation as a means of constructing built-in fault tolerant quantum logic gates [1–3]. This conjecture is based on the properties of the GP: as a geometric feature it is much less dependent on the details of the time evolution and may be less affected by uncontrolled fluctuations and therefore more robust against certain sources of perturbations.

There have been many proposals to extend the notion of GP for open systems as e.g. based on the state purification, quantum trajectories and quantum interferometry (kinematic approach). The last concept seems to be very promising. It can be applied to low-dimensional open systems for which the reduced dynamics is known. One of the most 'popular' guidelines for the description of reduced dynamics is the Kossakowski–Lindblad form of the master equations [4]. This approach has been successfully applied in reducing decoherence [5] or dynamics of quantum entanglement [6–9]. In this context the GPs gained by time-evolving quantum systems attract considerable attention. They are present either in adiabatically [10] or arbitrarily [11] evolving systems. The extension to the mixed states has been proposed first in [12] in a purely mathematical fashion and in [13] for unitary evolutions with a clear interferometric interpretation.

In the paper, we follow the kinematic approach [14] and consider an arbitrary reduced dynamics of a quantum system determined by the map

$$\rho(0) \to \rho(t) = \sum_i p_i(t)|w_i(t)\rangle\langle w_i(t)|, \tag{1}$$

where the spectral representation of the reduced density operator $\rho(t)$ is used: $p_i(t)$ and $|w_i(t)\rangle$ are eigenvalues and eigenvectors of the reduced density operator $\rho(t)$. The GP $\Phi(t)$ associated with such an evolution is defined by the expression [14]

$$\Phi(t) = \arg\left[\sum_i [p_i(0)p_i(t)]^{1/2}\langle w_i(0)|w_i(t)\rangle \exp\left(-\int_0^t \langle w_i(s)|\dot{w}_i(s)\rangle\, \mathrm{d}s\right)\right], \tag{2}$$

which in principle can be measurable [14].

The approach to the reduced dynamics is particularly convenient in the widely used Markovian approximation [4] which is justified in the weak coupling limit or in the singular coupling limit. However, the results obtained in the weak coupling limit cannot be extrapolated to the low temperature regime. Therefore, its applicability for solid-state devices operating at extremely low temperatures is problematic. Both the temperature of the environment [15] and the non-Markovian character of the reduced dynamics [16, 17] can radically influence the GPs. The key problem concerns the GP in systems coupled to real environments. The dissipation and/or pure dephasing caused by such environments is, in general, neither Markovian nor weak. In this paper, we study the GP in a system under such circumstances and investigate whether there are optimal regimes in which the GP weakly depends on the environment or is even completely independent. Such regimes would be the best from the point of view of the geometric quantum computation.

2. Model

We study the simplest open system consisting of a qubit S, formed by an arbitrary two-level system or spin-1/2 particle. We assume that there is no exchange energy with the environment (no energy dissipation). It is purely a dephasing coupling which is an irreversible process of information loss [18]. We assume that the Hamiltonian of the system takes the form [19]

$$H = \hbar\omega_0 S^z + \sum_{k=1}^{\infty} \hbar g_k\left(a_k^\dagger + a_k\right)S^z + \sum_{k=1}^{\infty} \hbar\omega_k a_k^\dagger a_k, \tag{3}$$

where the qubit is represented by the spin-1/2 operator S. The environment is modelled by harmonic oscillators, a_k and $a_k^\dagger$ are the annihilation and creation Bose operators, g_k is the strength of coupling between the qubit and the kth mode of the environment. Such a model may serve as a component of a simple quantum register [18].

The reduced dynamics of the qubit can *exactly* be determined for arbitrary model parameters provided the initial state of the total system $\varrho(0)$ can be factorized into the qubit state $\rho(0)$ and the state of the environment ρ_{env}, namely, $\varrho(0) = \rho(0) \otimes \rho_{\mathrm{env}}$, where ρ_{env} is the Gibbs state of temperature T. This exact reduced dynamics was derived and analysed for the first time in [19].

The statistical operator $\rho(t)$ of the qubit at time $t > 0$ has the form (cf equation (5.19) in [19])

$$\rho(t) = C_1(t)\rho(0) + 2C_2(t)[S^z, \rho(0)] + 4C_3(t)S^z\rho(0)S^z \tag{4}$$

for the arbitrary initial statistical operator $\rho(0)$. The functions

$$C_1(t) = \tfrac{1}{2}[1 + A(t)\cos\omega_0 t],$$
$$C_2(t) = \tfrac{1}{2}\mathrm{i}A(t)\sin\omega_0 t, \tag{5}$$
$$C_3(t) = \tfrac{1}{2}[1 - A(t)\cos\omega_0 t],$$

where

$$A(t) = \exp[-f(t)], \qquad f(t) = \int_0^\infty \mathrm{d}\omega \frac{J(\omega)}{\omega^2}\coth(\hbar\omega\beta_0/2)[1 - \cos\omega t], \tag{6}$$

where $\beta_0 = 1/kT$, k is the Boltzmann constant. The corresponding generator of the reduced dynamics is of the Kossakowski–Lindblad form [19] and hence complete positivity is preserved [4]. The environment is characterized by the frequency spectrum $J(\omega) = \sum_k g_k^2\delta(\omega - \omega_k)$, which in the thermodynamic limit for the environment is assumed to take the form [20, 22]

$$J(\omega) = \lambda_\mu \omega^{1+\mu}\exp(-\omega/\omega_c), \qquad \mu > -1. \tag{7}$$

The cut-off ω_c determines the largest energy scale of the environment (it removes possible divergences at high frequencies; in all calculations presented below we assume $\omega_c/\omega_0 = 10^3$) and λ_μ is the coupling strength of the qubit and environment. The spectral exponent μ characterizes the low frequency properties of the environment and defines its various types. According to the classification proposed in [20], the environment is called subohmic for $\mu \in (-1, 0)$, ohmic for $\mu = 0$ and superohmic for $\mu \in (0, \infty)$. This classification shall be reflected in the dynamical properties of the GP gained in a cyclic evolution. We study the GP of the system initially prepared in the pure state

$$\rho(0) = |\psi(\theta)\rangle\langle\psi(\theta)| \tag{8}$$

with

$$|\psi(\theta)\rangle = \cos(\theta/2)|1\rangle + \sin(\theta/2)|-1\rangle, \tag{9}$$

where $\{|1\rangle, |-1\rangle\}$ is the standard qubit basis. In this basis, the statistical operator (4) takes the matrix form

$$\rho(t) = \begin{pmatrix} \cos^2(\theta/2) & (1/2)A(t)\,\mathrm{e}^{\mathrm{i}\omega_0 t}\sin\theta \\ (1/2)A(t)\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\sin\theta & \sin^2(\theta/2) \end{pmatrix}. \tag{10}$$

The eigenvalues and eigenvectors of this matrix can easily be calculated and explicit expression for the GP in equation (2) can be obtained, cf equation (8) in [17] or equation (9) in [16]. The evolution of the spin coupled to the environment is not periodic. However, since the evolution of the uncoupled spin is cyclic with the period $T = 2\pi/\omega_0$, we calculate the GP at this time, i.e. $\Phi(2\pi/\omega_0)$. The results of our study can be compared to the GP of a 'noiseless' system (with $\lambda = 0$) of the same initial preparation given by [21]

$$\Phi_0 = \pi[\cos(\theta) - 1]. \tag{11}$$

It is the common wisdom that the coupling to the environment reduces the GPs which remain significant only below a certain time related to the dephasing time [15–17]. The dependence of the phase both on the initial preparation of the system and low frequency spectral properties of the environment, coded in $J(\omega)$, is less obvious.

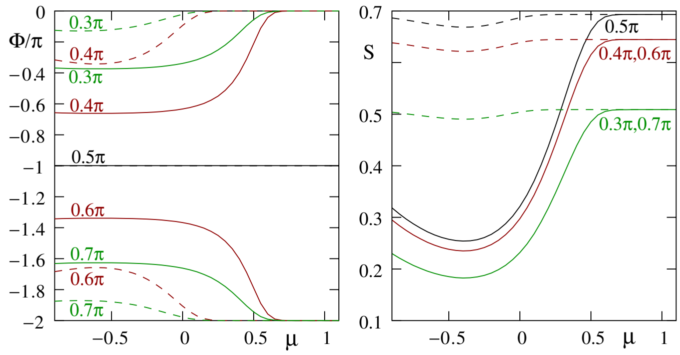

Figure 1. The geometric phase $\Phi = \Phi(2\pi/\omega_0)$ (left panel) and the von-Neumann entropy S (right panel) versus the spectral exponent μ for zero temperature, $T = 0$, and various initial states parametrized by the angle, $\theta = 0.3\pi, \ldots, 0.7\pi$. The solid (dashed) line corresponds to the dimensionless coupling strength $\lambda = \lambda_\mu \omega_0^\mu = 0.1$ ($\lambda = 1$).

3. $T = 0$ limit

In the $T = 0$ limit, the quantum-mechanical properties are most transparent because the 'classical' sources of dissipation, decoherence and dephasing are frozen. The zero-temperature fluctuations are unavoidable due to vacuum fluctuations of the environment. It results in a non-unitary evolution with the dephasing function [9]

$$K(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} = \lambda \frac{\Gamma(1+\mu)\omega_c^{1+\mu}}{\left(1+\omega_c^2 t^2\right)^{(1+\mu)/2}} \sin[(1+\mu)\arctan(\omega_c t)]. \tag{12}$$

Let us recall that for the model of Markovian dynamics the dephasing function is time independent, $K_{\text{markov}}(t) = \text{const}$ [4]. However, the Markovian approximation is justified only in the regimes where the energy scale of the coupling to the environment is significantly smaller than any other energy scale in the system [4, 22]. In this sense, the Markovian approximation of the real reservoir at $T = 0$ suffers from inconsistency and can be regarded as a formal 'toy' only. The real dissipation and the Markovian toy are essentially different. In particular, $\lim_{t\to\infty} K(t) = 0$ what seems to be crucial for various effects reported below.

Let us analyse the influence of various factors on the GP at $T = 0$. From the exact results presented in figures 1 and 2, it follows that for the weak coupling to environment (here the rescaled dimensionless coupling strength $\lambda = \lambda_\mu \omega_0^\mu = 0.1$) the phase weakly depends on the spectral exponent μ in the subohmic and weakly superohmic environment. The interval where Φ is almost constant as a function of μ depends on the initial state of the qubit, i.e. on the angle θ. When θ approaches $\pi/2$, Φ becomes insensitive to the spectral exponent even for large values of μ. In the limit $\theta \to \pi/2$, the size of plateau in $\Phi(\mu)$ becomes infinite. Exactly for $\theta = \pi/2$ the phase depends neither on μ nor on λ. So, the initial state $|\psi\rangle = (\sqrt{2}/2)[|1\rangle + |-1\rangle]$ is the best in the sense of robustness against the dephasing environment. The 'stiffness' of the phase at $\theta = \pi/2$ can be inferred from the analytic results obtained in [14] for a Markovian approximation and arbitrary coupling. This feature has also been derived in [17] for ohmic and superohmic environment within the perturbative approach with respect to λ. Namely, the first-order correction is proportional to $\cos\theta$ and vanishes for $\theta = \pi/2$. To prove it for an arbitrary coupling strength λ let us note that for $\theta = \pi/2$ the mean value $\langle S^z(t)\rangle = 0$ for any time. As a result, the instantaneous eigenvectors of $\rho(t)$ are independent of the environment. On the other hand, the corresponding eigenvalues do

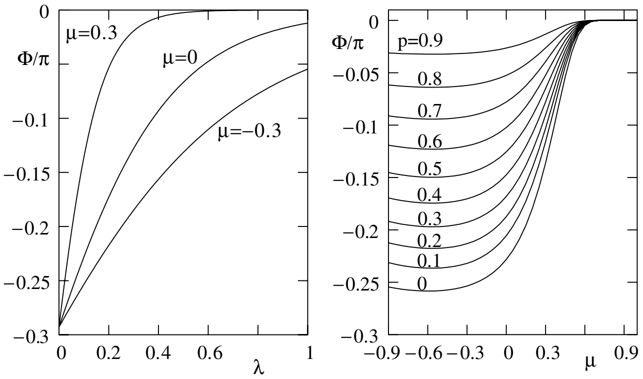

Figure 2. Left panel: the geometric phase $\Phi = \Phi(2\pi/\omega_0)$ versus the coupling strength λ for $T = 0, \theta = \pi/4$ and three values of μ. Right panel: the geometric phase for initially depolarized state with ten values of the probability p, for $T = 0, \theta = \pi/4$ and $\lambda = 0.1$.

depend on the environment via the function $f(t)$ defined in equation (6). This property of the spectral decomposition of $\rho(t)$ holds true for any state provided $\langle S^z(0)\rangle = 0$. In consequence, it explains the different influence of the environment on the GP and entropy as can be inferred from figure 1. One can observe that the phase takes an extremal (minimal or maximal) value at the optimal subohmic environment characterized by the exponent $\mu < 0$. The extremum is more pronounced for stronger coupling λ, cf the dashed lines in the left panel of figure 1. The influence of the coupling strength on the phase is depicted in figure 2. The superohmic environment is most destructive for the GP. In particular, $\Phi(\lambda)$ decays faster for larger values of μ. It can be inferred from the left panel in figure 2 that applicability of the linear expansion strongly depends on the spectral properties of the environment. In the subohmic case, the linear approximation for $\Phi(\lambda)$ holds even for a relatively large λ, whereas in the superohmic case this linear dependence breaks down already for $\lambda \ll 1$.

In some regimes of μ, there is a direct relation between the decay of the GP and the mixedness of the qubit state at $t = 2\pi/\omega_0$. The latter property is quantified by the von Neumann entropy shown in the right panel of figure 1. The decay of $\Phi(\mu)$ is accompanied by the increase of the entropy almost to its maximal value. However, it is true only above a certain value of μ for which the entropy is minimal. An explanation of this behaviour is possible in terms of properties of the dephasing function $K(t)$ whose short-time properties are governed by the presence of a strong peak [9]. The amplitude of this peak increases when μ changes from the subohmic to the superohmic regime. This peak is responsible for a rapid dephasing of the qubit. Its influence on the GP is, in some sense, complementary to the recently reported persistent entanglement in a two-qubit system [9].

The specific role of the initial preparation becomes most transparent from the comparison of the both panels presented in figure 1. As discussed above, the GP is insensitive to the environment for $\theta = \pi/2$. However, it does *not* mean that the system becomes uncoupled from the environment. The corresponding von Neumann entropy becomes maximal for $\theta = \pi/2$, which clearly indicates that the qubit is affected by the bath.

4. Non-zero temperature and noisy initial states

In this section, we address the natural question to what extent the results discussed in the previous section are preserved in real systems either operating at non-zero temperature or

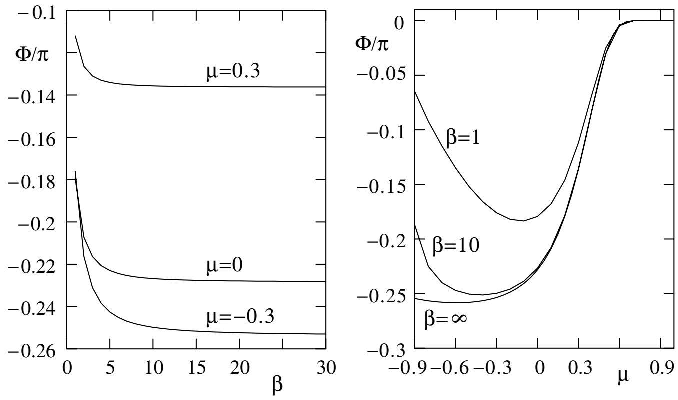

Figure 3. Left panel: the geometric phase $\Phi = \Phi(2\pi/\omega_0)$ versus the inverse temperature $\hbar\omega_0/kT$ for $\theta = \pi/4$ and three selected values of the spectral exponent μ. Right panel: the geometric phase $\Phi = \Phi(2\pi/\omega_0)$ versus the spectral exponent μ, for the fixed initial state determined by $\theta = \pi/4$, three values of the inverse temperature $\beta = \hbar\omega_0/kT = 1, 10, \infty$ and the coupling constant $\lambda = 0.1$.

suffering from imperfect initial preparation. The impact of the temperature can be inferred from figure 3, where we present the GP for the system initially prepared in the state defined by equation (9) with $\theta = \pi/4$. It seems that the value of μ at which the phase is lost does not depend on temperature. It is also interesting that there is a subohmic environment for which the GP is extremal. However, the corresponding value of μ depends on temperature.

The effect of imperfect initial preparation is modelled by a depolarized initial state [23]:

$$|\psi(\theta)\rangle\langle\psi(\theta)| \longrightarrow (1-p)|\psi(\theta)\rangle\langle\psi(\theta)| + \frac{p}{2}I \tag{13}$$

where the unit matrix is denoted by I and $p \in [0, 1]$. The results for $\theta = \pi/4$ are presented in figure 2. Again, the critical value of μ at which the phase is lost does not depend on p, except the trivial case $p = 1$.

Similarly to the zero-temperature case, the phase-coherence decay is essentially related to the spectral properties of the bath. However, for $\theta = \pi/2$ the phase depends neither on temperature nor the initial depolarization quantified by p. These results originate from the same arguments, which were discussed in the previous section. As the investigated depolarizing channel is very specific, the results of this section cannot be straightforwardly extended to other types of the initial noise.

5. Summary

We have presented exact results for the geometric phase of the qubit dephasingly coupled to the environment. We have shown that there is a certain value of μ that is critical for the phase-coherence of the system. This particular value of μ hardly depends on temperature and a model of initial noise. Nevertheless, this value always corresponds to the superohmic regime. When μ exceeds this threshold the phase gained in the evolution of the qubit vanishes. This effect strongly depends on the choice of initial preparation of the system. If the qubit is initially prepared in the eigenstate of the S^x, the phase is insensitive to both the strength of coupling to the environment and the temperature. However, such a qubit is not decoupled from the environment since its entropy evolves in time. It is due to a specific form of the spectral decomposition of the reduced density matrix of the dephasingly evolving spin with vanishing $\langle S^z \rangle$.

Our analysis shows that the optimal robustness of the phase against the environmental noise occurs for a subohmic bath. In contrast, the superohmic environments are almost always destructive for the phase. The results presented in the paper can potentially be important for the design of 'holonomic' quantum computers based on flux qubits [24, 25] with effective environmental engineering [26].

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